Technical Comments

Comment on "Modal Coupling in Lightly Damped Structures"

Alexander H. Flax*
Institute for Defense Analyses, Arlington, Va.

HASSELMAN[1] has investigated criteria for modal coupling of lightly damped structures. His results are obtainable more simply and directly from the classical perturbation theory for lightly damped structures given by Rayleigh.[2] In addition, Rayleigh's derivation makes the mathematical treatment of orders of magnitude more explicit and obvious and permits computation of the frequency corrections for light damping to the second order in damping perturbation.

Rayleigh's theory begins with an unperturbed system that is undamped and is characterized by kinetic energy T and potential energy V in terms of the normal modes of vibration as generalized coordinates with amplitude q_n. These are, by definition, uncoupled. Thus,

$$T = ½\, a_1 \dot{q}_1^2 + ½\, a_2 \dot{q}_2^2 + \cdots \tag{1a}$$

$$V = ½\, c_1 q_1^2 + c_2 q_2^2 + \cdots \tag{1b}$$

where, as is apparent, a_n and c_n are, respectively, the generalized mass and stiffness of the nth normal mode.

The perturbation comprises small damping terms that may couple the modes together. Thus damping is introduced in terms of the Rayleigh dissipation function F, where

$$F = ½\, b_{11} \dot{q}_1^2 + ½\, b_{22} \dot{q}_2^2 + \cdots + b_{12} \dot{q}_1 \dot{q}_2 + b_{13} \dot{q}_1 \dot{q}_3 + \cdots \tag{2}$$

Using Lagrange's equations with the dissipation terms and supposing variations of all time coordinates as e^{pt} leads to a set of equations equal in number to the number of degrees of freedom. The nth equation of this set is

$$[p^2 a_m + b_m p + c_m] q_m + p \Sigma b_{mn} q_n = 0 \tag{3}$$

Considering the damping as small and taking the unperturbed natural frequency $p_{k0} = i\omega_{k0}$, the first-order correction to the kth normal mode is obtained as the ratio of the nth modal coordinate to the kth:

$$\frac{q_n}{q_k} = -\frac{b_{kn} p_{k0}}{a_n (p_{n0}^2 - p_{k0}^2)} \tag{4}$$

This expression leads to criteria for modal coupling identical to those arrived at by Hasselman.

In addition, Rayleigh gives the equation for the computation of the natural frequencies (more properly in this case, eigenvalues that are complex for the damped case) to the second order in the magnitude of damping as

$$a_k p_k^2 + b_{kk} p_k + c_k + \sum_{n \neq k} \frac{p_{k0}^2 b_{kn}^2}{a_n (p_{n0}^2 - p_{k0}^2)} = 0 \tag{5}$$

Introducing the damping ratio of the kth mode as $\zeta_k = b_{kk}/2a_k \omega_{k0}$ and noting that $p_{k0} = i\omega_{k0}$, Eq. (4) becomes

$$\left| \frac{q_n}{q_k} \right| = \frac{2\,\zeta_k}{|(\omega_{k0}^2/\omega_{n0}^2) - 1|} \left(\frac{\xi_{kn}}{\xi_{kk}} \right) \tag{6}$$

where $\xi_{kk} = 2\zeta_k \omega_{k0}$ and $\xi_{kn} = b_{kn}/a_n$ to correspond to Hasselman's definitions. Here the purely imaginary phase of Eq. (4) has been suppressed by converting to absolute values. If, in Eq. (6), the requirement is set that $q_n/q_k \ll 1$, the result is exactly the inequality given by Hasselman's Eq. (15).

References

[1] Hasselman, T.K., "Modal Coupling in Lightly Damped Structures," *AIAA Journal*, Vol. 14, Nov. 1976, pp. 1627-1628.

[2] Rayleigh, B., *The Theory of Sound*, Dover, New York, 1945, Vol. 1, pp. 136-137.

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*President. Fellow AIAA.

Reply by Author to A. H. Flax

T. K. Hasselman*
J. H. Wiggins Company, Redondo Beach, Calif.

THE comment by Flax suggests an alternative criterion for neglecting the off-diagonal coupling terms in the modal damping matrix of a lightly damped structure. It first should be recognized as being different from the one presented in inequality (8) of Ref. 1. In particular, Flax implies that, whenever

$$|q_k/q_j| \ll 1 \tag{1}$$

the modal coupling term ξ_{kj} may be neglected. This inequality (1) leads to the criterion

$$\frac{2\zeta_j}{[(\beta^2-1)^2 + 4\zeta_k^2 \beta^2]^{½}} \left| \frac{\xi_{kj}}{\xi_{jj}} \right| \ll 1 \tag{2}$$

whenever the kth equation (row) of the set

$$\{([\omega^2] - \Omega^2 [I]) + i\Omega[\xi]\}\{q(i\Omega)\} = \{0\} \tag{3}$$

is used with $\Omega \equiv \omega_j$; $\beta = \omega_k/\omega_j$.

Reference 1 asserts that, whenever the inequality

$$|\{e\}_k^T [\tilde{Z}_n(i\Omega)]\{e\}_j| \ll 1 \tag{4}$$

is true, the modal coupling term ξ_{kj} may be neglected. This leads to the criterion

$$\left\{ \frac{2\zeta_j}{[(\beta^2-1)^2 + 4\zeta_k^2 \beta^2]^{½}} \right\}^{½} \left| \frac{\xi_{kj}}{\xi_{jj}} \right| \ll 1 \tag{5}$$

which is clearly different from Flax's inequality (2).

The question is, which criterion (if either) is proper? If one were to choose $\Omega \equiv \omega_k$ in Eq. (3) for deriving the ratio q_k/q_j, inequality (1) would lead to

$$\frac{\zeta_j}{\zeta_k \beta} \left| \frac{\xi_{kj}}{\xi_{jj}} \right| \ll 1 \tag{6}$$

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*Manager, Engineering Mechanics Department. Member AIAA.